TEST CODE **02134020**

FORM TP 2011231

MAY/JUNE 2011

CARIBBEAN EXAMINATIONS COUNCIL

ADVANCED PROFICIENCY EXAMINATION

PURE MATHEMATICS

UNIT 1 – PAPER 02

ALGEBRA, GEOMETRY AND CALCULUS

2 ½ hours

10 MAY 2011 (p.m.)

This examination paper consists of **THREE** sections: Module 1, Module 2 and Module 3.

Each section consists of 2 questions.
The maximum mark for each Module is 50.
The maximum mark for this examination is 150.
This examination consists of 7 printed pages.

INSTRUCTIONS TO CANDIDATES

1. **DO NOT** open this examination paper until instructed to do so.
2. Answer **ALL** questions from the **THREE** sections.
3. Write your solutions, with full working, in the answer booklet provided.
4. Unless otherwise stated in the question, any numerical answer that is not exact **MUST** be written correct to three significant figures.

Examination Materials Permitted

Graph paper (provided)
Mathematical formulae and tables (provided) – **Revised 2010**
Mathematical instruments
Silent, non-programmable, electronic calculator

Copyright © 2010 Caribbean Examinations Council
All rights reserved.

SECTION A (Module 1)

Answer BOTH questions.

1. (a) Without using calculators, find the exact value of

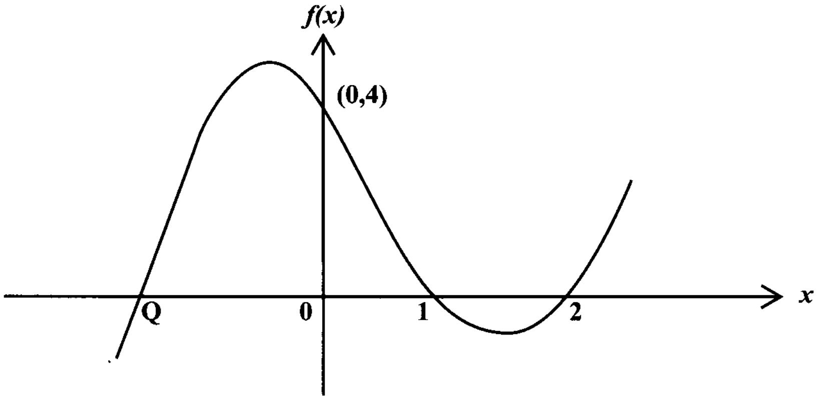
(i) $(\sqrt{75} + \sqrt{12})^2 - (\sqrt{75} - \sqrt{12})^2$ **[3 marks]**

(ii) $27^{\frac{1}{4}} \times 9^{\frac{3}{8}} \times 81^{\frac{1}{8}}$. **[3 marks]**

(b) The diagram below, **not drawn to scale**, represents a segment of the graph of the function

$$f(x) = x^3 + mx^2 + nx + p$$

where m, n and p are constants.



Find

(i) the value of p **[2 marks]**

(ii) the values of m and n **[4 marks]**

(iii) the x-coordinate of the point **Q**. **[2 marks]**

(c) (i) By substituting $y = \log_2 x$, or otherwise, solve, for x, the equation

$$\sqrt{\log_2 x} = \log_2 \sqrt{x}\ .$$ **[6 marks]**

(ii) Solve, for real values of x , the inequality

$$x^2 - |\,x\,| - 12 < 0.$$ **[5 marks]**

Total 25 marks

GO ON TO THE NEXT PAGE

2. (a) The quadratic equation $x^2 - px + 24 = 0$, $p \in \mathbf{R}$, has roots α and β.

(i) Express in terms of p

a) $\alpha + \beta$ **[1 mark]**

b) $\alpha^2 + \beta^2$. **[4 marks]**

(ii) Given that $\alpha^2 + \beta^2 = 33$, find the possible values of p. **[3 marks]**

(b) The function $f(x)$ has the property that

$$f(2x + 3) = 2f(x) + 3,\ x \in \mathbf{R}.$$

If $f(0) = 6$, find the value of

(i) $f(3)$ **[4 marks]**

(ii) $f(9)$ **[2 marks]**

(iii) $f(-3)$. **[3 marks]**

(c) Prove that the product of any two consecutive integers k and $k + 1$ is an even integer. **[2 marks]**

(d) Prove, by mathematical induction, that $n\ (n^2 + 5)$ is divisible by 6 for all positive integers n. **[6 marks]**

Total 25 marks

SECTION B (Module 2)

Answer BOTH questions.

3. (a) (i) Let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$ with $|\mathbf{a}| = 13$ and $|\mathbf{b}| = 10$. Find the value of $(\mathbf{a} + \mathbf{b}) . (\mathbf{a} - \mathbf{b})$. **[5 marks]**

(ii) If $2\mathbf{b} - \mathbf{a} = 11\mathbf{i}$, determine the possible values of **a** and **b**. **[5 marks]**

(b) The line L has equation $x - y + 1 = 0$ and the circle C has equation $x^2 + y^2 - 2y - 15 = 0$.

(i) Show that L passes through the centre of C. **[2 marks]**

(ii) If L intersects C at P and Q, determine the coordinates of P and Q. **[3 marks]**

(iii) Find the constants a, b and c such that $x = b + a \cos\theta$ and $y = c + a \sin\theta$ are parametric equations (in parameter θ) of C. **[3 marks]**

(iv) Another circle C_2, with the same radius as C, touches L at the centre of C. Find the possible equations of C_2.

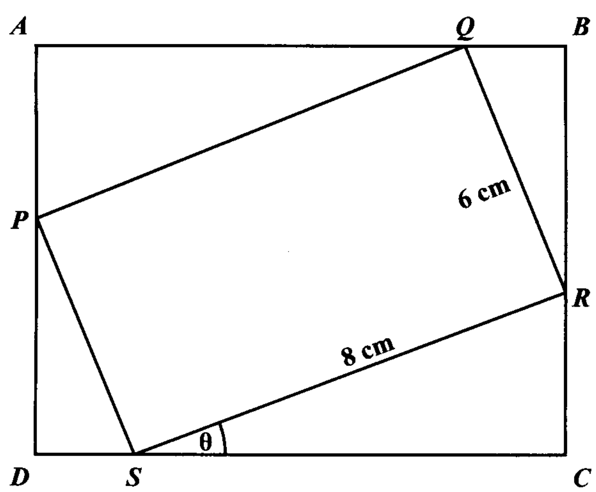
[7 marks]

Total 25 marks

4. (a) By using $x = \cos^2\theta$, or otherwise, find all values of the angle θ such that

$$8 \cos^4\theta - 10 \cos^2\theta + 3 = 0, 0 \leq \theta \leq \pi.$$ **[6 marks]**

(b) The diagram below, **not drawn to scale**, shows a rectangle $PQRS$ with sides 6 cm and 8 cm inscribed in another rectangle $ABCD$.



GO ON TO THE NEXT PAGE

(i) The angle that SR makes with DC is θ. Find, in terms of θ, the length of the side BC. **[2 marks]**

(ii) Find the value of θ if $|BC| = 7$ cm. **[5 marks]**

(iii) Is 15 a possible value for $|BC|$? **Give a reason for your answer.** **[2 marks]**

(c) (i) Show that $\dfrac{1 - \cos 2\theta}{\sin 2\theta} = \tan \theta$. **[3 marks]**

(ii) Hence, show that

a) $\dfrac{1 - \cos 4\theta}{\sin 4\theta} = \tan 2\theta$. **[3 marks]**

b) $\dfrac{1 - \cos 6\theta}{\sin 6\theta} = \tan 3\theta$. **[2 marks]**

(iii) Using the results in (c) (i) and (ii) above, evaluate

$$\sum_{r=1}^{n} (\tan r\theta \sin 2r\theta + \cos 2r\theta)$$

where n is a positive integer. **[2 marks]**

Total 25 marks

GO ON TO THE NEXT PAGE

SECTION C (Module 3)

Answer BOTH questions.

5. (a) Find $\lim_{x \to -2} \dfrac{x^2 + 5x + 6}{x^2 - x - 6}$. **[4 marks]**

(b) The function f on **R** is defined by

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x \geq 2 \\ bx + 1 & \text{if } x < 2. \end{cases}$$

Determine

(i) $f(2)$ **[2 marks]**

(ii) $\lim_{x \to 2^+} f(x)$ **[2 marks]**

(iii) $\lim_{x \to 2^-} f(x)$ in terms of the constant b **[2 marks]**

(iv) the value of b such that f is continuous at $x = 2$. **[4 marks]**

(c) The curve $y = px^3 + qx^2 + 3x + 2$ passes through the point $T(1, 2)$ and its gradient at T is 7. The line $x = 1$ cuts the x-axis at M, and the normal to the curve at T cuts the x-axis at N.

Find

(i) the values of the constants p and q **[6 marks]**

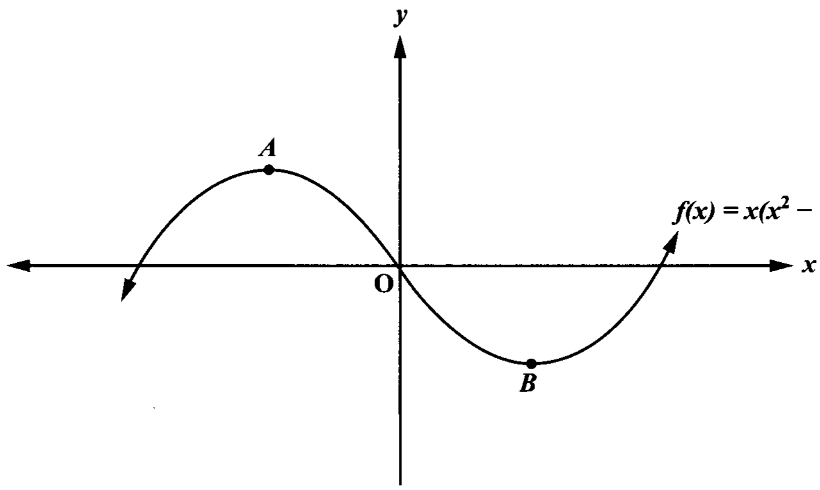
(ii) the equation of the normal to the curve at T **[3 marks]**

(iii) the length of MN.

[2 marks]

Total 25 marks

6. (a) The diagram below, **not drawn to scale**, is a sketch of the section of the function $f(x) = x(x^2 - 12)$ which passes through the origin O. A and B are stationary points on the curve.



Find

(i) the coordinates of each of the stationary points A and B **[8 marks]**

(ii) the equation of the normal to the curve $f(x) = x(x^2 - 12)$ at the origin, O **[2 marks]**

(iii) the area between the curve and the positive x-axis. **[6 marks]**

(b) (i) Use the result

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx,\ a > 0,$$

to show that $\int_0^\pi x \sin x\,dx = \int_0^\pi (\pi - x)\sin x\,dx$. **[2 marks]**

(ii) Hence, show that

a) $\int_0^\pi x \sin x\,dx = \pi \int_0^\pi \sin x\,dx - \int_0^\pi x \sin x\,dx$ **[2 marks]**

b) $\int_0^\pi x \sin x\,dx = \pi.$ **[5 marks]**

Total 25 marks

END OF TEST